Recap

Earlier

Linear system: $\dot{x} = Ax + Bu$, $y = Cx + Du$

Input: $u(t) = e^{st}$ (complex $s \neq \lambda(A)$).

Steady-state output: $y_{ss}(t) = \underbrace{\left[D + C(sI - A)^{-1}B\right]}_{\text{transfer function}} u(t)$

Last lecture

For an ODE with output y and input u of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y$$
$$= b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u.$$

the transfer function is

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

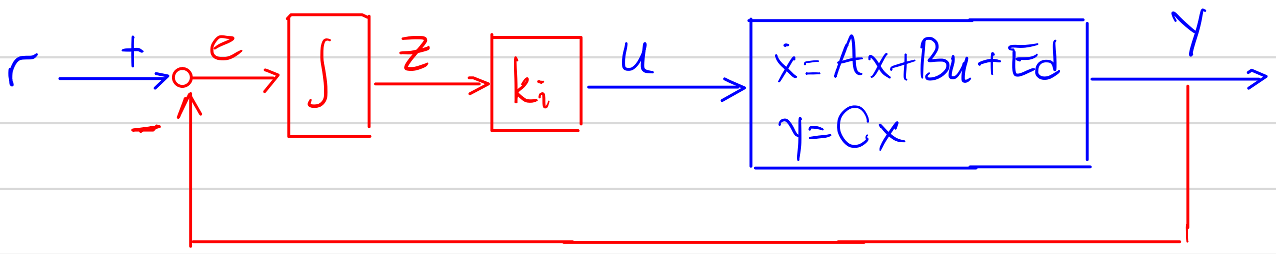
- Poles of G = roots of $a(s) = 0$
- Zeros of G = roots of $b(s) = 0$
- "Poles of G = eigenvalues of A"

Today: PID controllers.

Relevant parts from the book: Section 10.1 (Basic Control Functions)

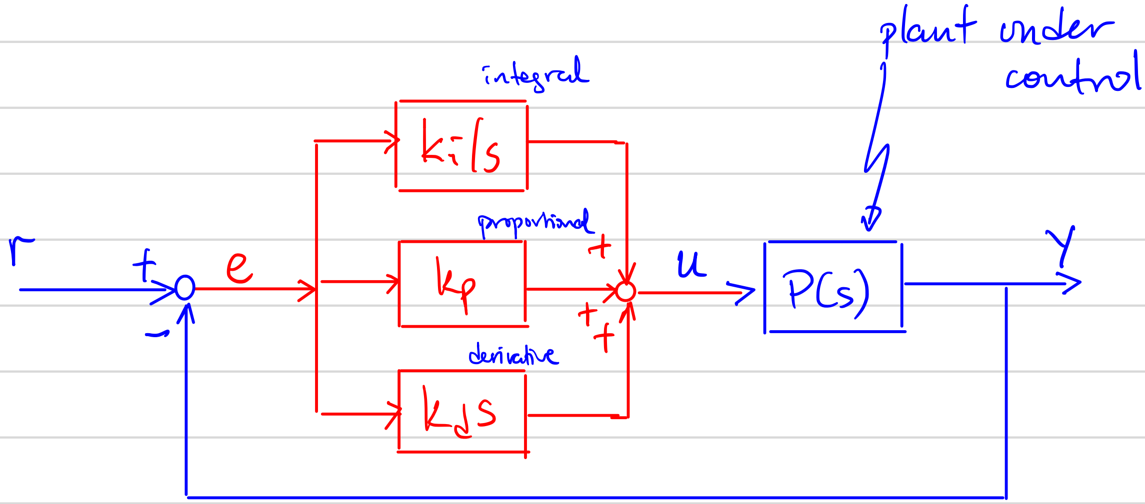
Proportional-integral-derivative (PID) control

Recall the control structure from the previous time we used integral action.

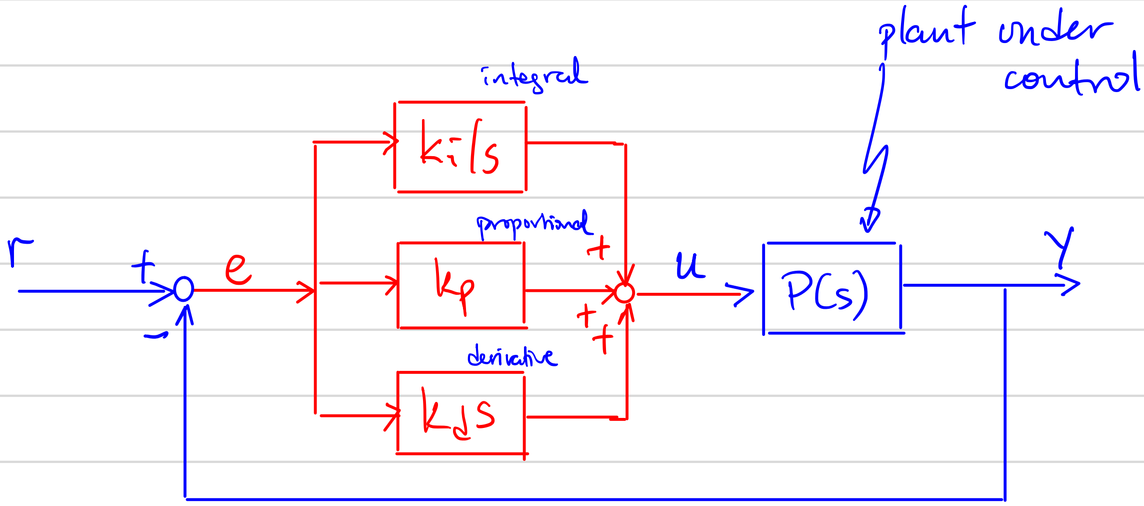


Now, we will work with transfer function representation of the system and controller.

Additionally, we will extend the controller structure.



Proportional-integral-derivative (PID) control



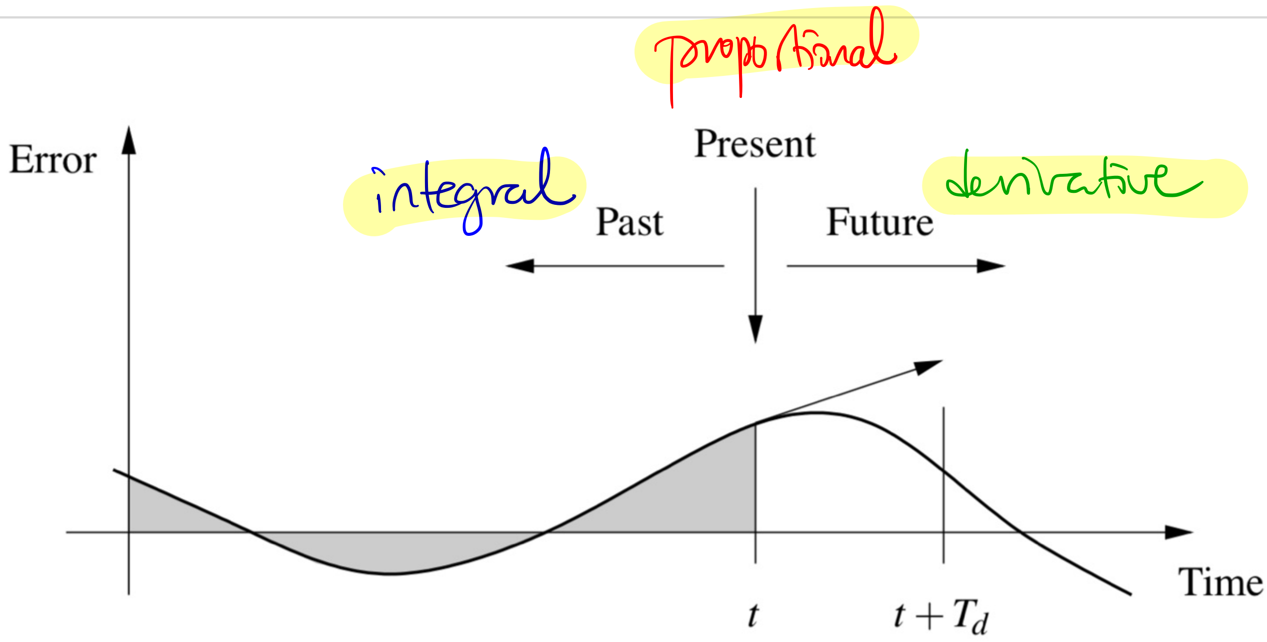
$e(t) = r(t) - y(t)$: error signal

$$u(t) = \underbrace{k_p e(t)}_{\text{proportional}} + \underbrace{k_i \int_0^t e(\tau)\, d\tau}_{\text{integral}} + \underbrace{k_d \frac{de(t)}{dt}}_{\text{derivative}}$$

Transfer function from e to u:

$$C(s) = k_p + k_i \frac{1}{s} + k_d s$$

Another interpretation of P, I and D terms

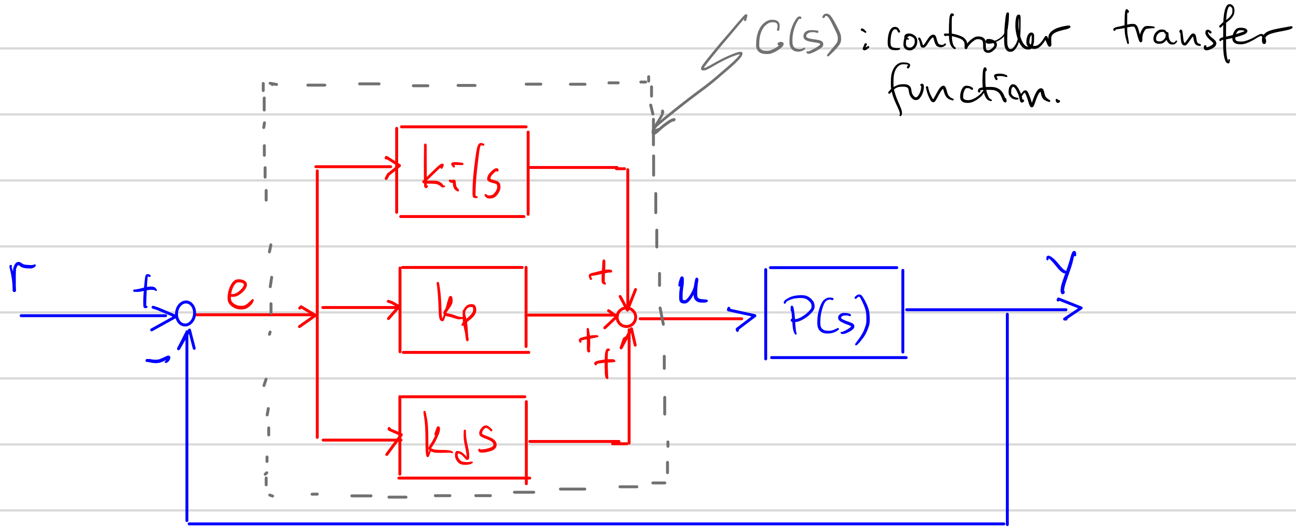


- Proportional part depends on the instanteneous error value.

- Integral part is based on the integral of the error upto time t.

- Derivative part provides an estimate of the growth of the error.

Closed-loop transfer function with a PID controller.



The transfer function from r to y:

$$G_{ry}(s) = \frac{P(s)\,C(s)}{1+P(s)\,C(s)}.$$

- Also, recall that the steady-state gain for a stable system modeled by a transfer function $H(s)$ under step input is $H(0)$.

First, consider pure proportional feedback (i.e., $k_i = 0$ and $k_d = 0$).

The steady-state output due to a <u>unit</u> step reference input is equal to

$$y_{ss} = G_{ry}(0) \cdot 1 = \frac{C(0)P(0)}{1 + C(0)P(0)}$$

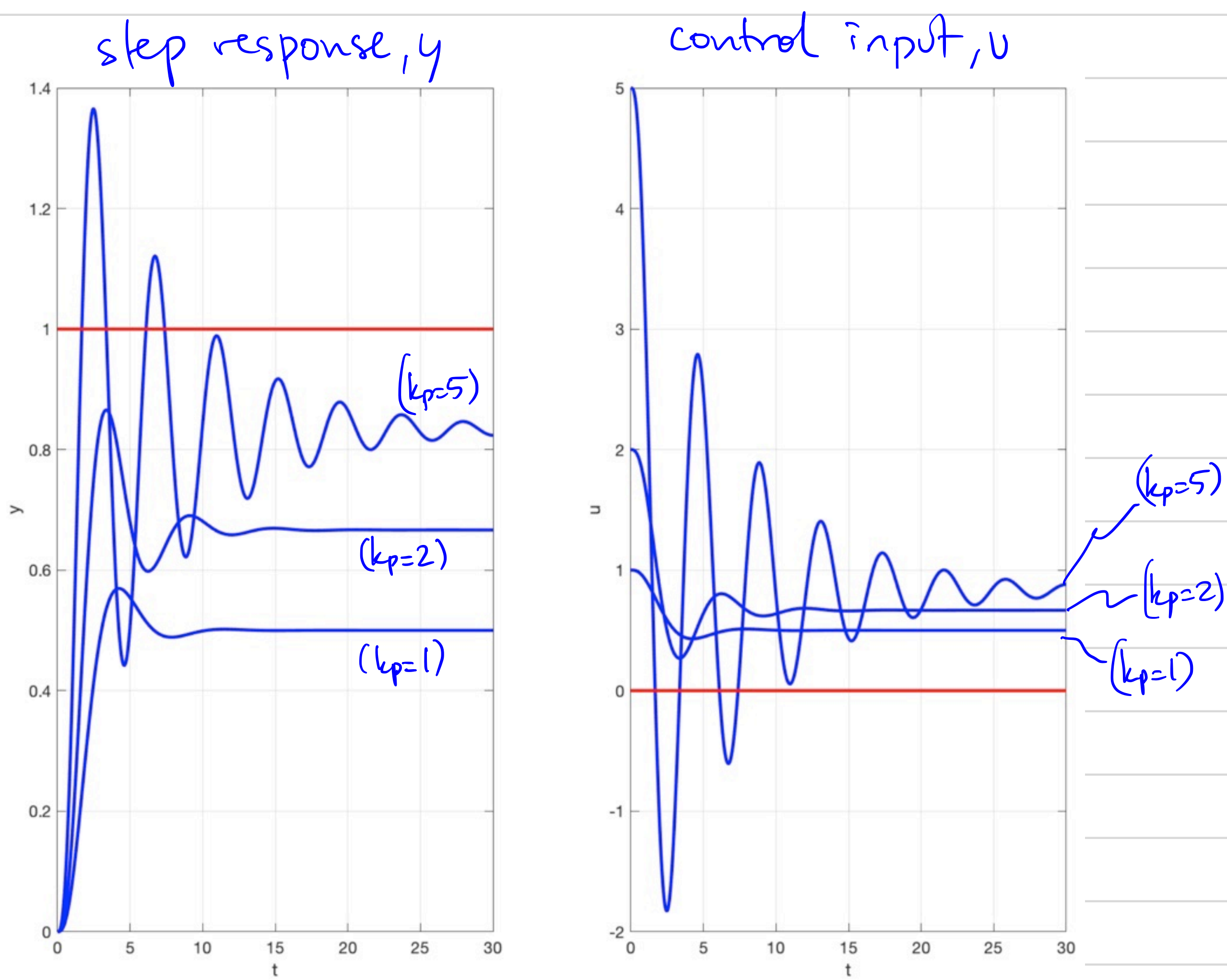
$$C(s) = k_p \Rightarrow C(0) = k_p$$

$$y_{ss} = \frac{k_p P(0)}{1 + k_p P(0)}$$

One can adjust k_p to make y_{ss} as close to 1 as possible.

Example (with P-term only)

$$P(s) = \frac{1}{(s+1)^3}$$



Increase kp:

- Reduces the steady-state error
- But, also introduces (larger) oscillations and overshoot.

Introduce integral feedback $C(s) = k_p + k_i \frac{1}{s}$

$$G_{ry}(s) = \frac{\left(k_p + \frac{k_i}{s}\right) P(s)}{1 + P(s)\left(k_p + \frac{k_i}{s}\right)}$$

$$= \frac{(k_p s + k_i)\frac{1}{s} P(s)}{s + P(s)(k_p s + k_i)\frac{1}{s}} = \frac{(k_p s + k_i) P(s)}{s + P(s)(k_p s + k_i)}$$

The steady-state output due to unit step reference input:

$$G_{ry}(0) = \frac{k_i P(0)}{k_i P(0)} = 1$$

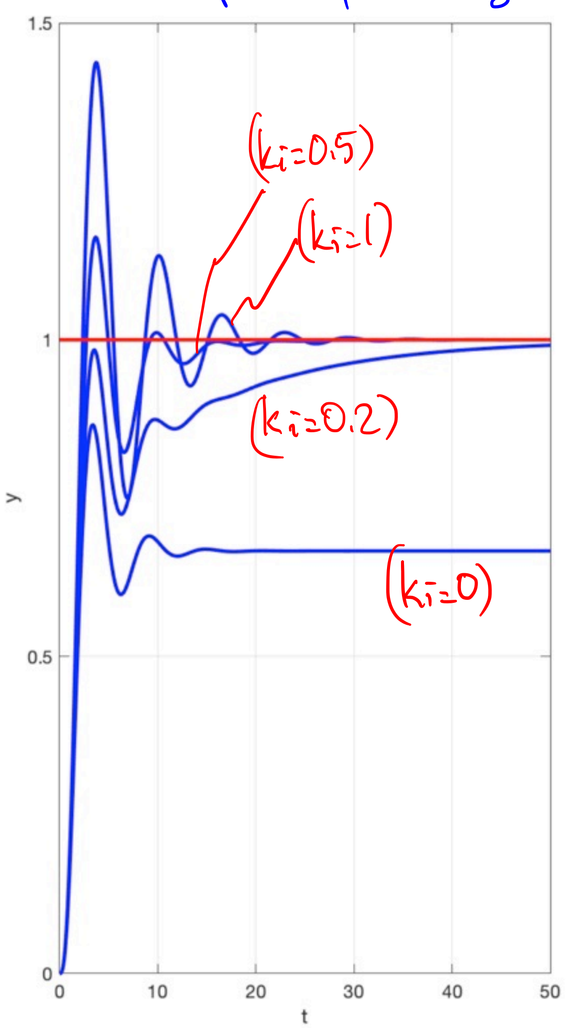
- Perfect reference tracking
- Independent from the plant parameters.

Recall: This is an important result. But, it is not news for us. We already had seen this fact. We just derived in a different way now.

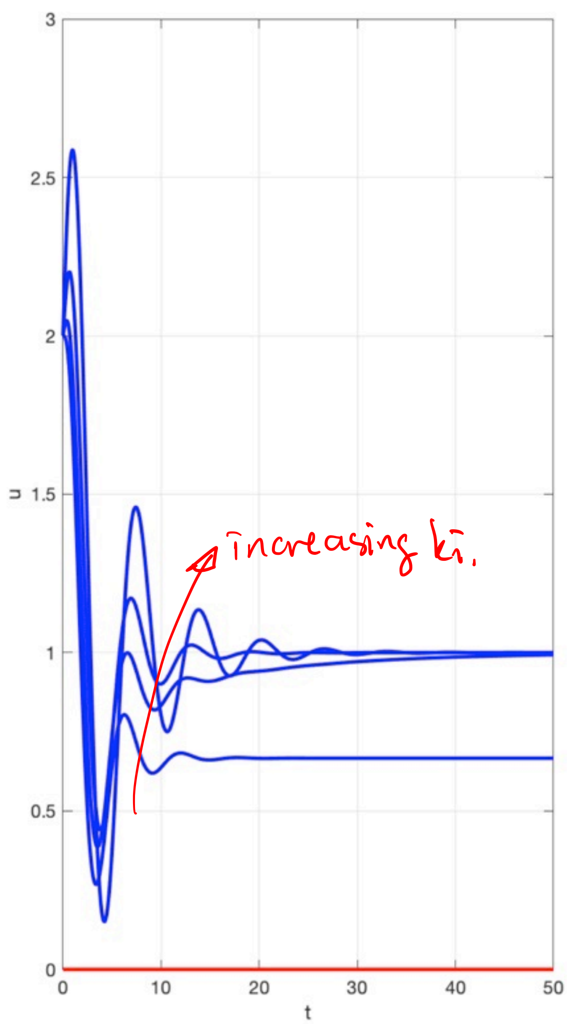
Example (with P and I terms)

Same plant as before. Fixed $k_p = 2$. Vary k_i.

step response, y

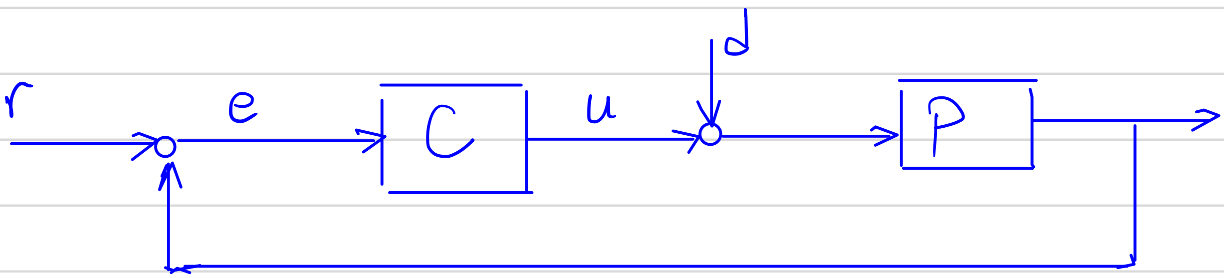


control input



- zero k_i $\Rightarrow$ non-zero steady-state error
- Nonzero k_i $\Rightarrow$ zero steady-state error
- As k_i increases, the approach to the steady-state output is faster.
- As k_i increases, the system becomes more oscillatory.

Another useful property of integral action:
disturbance attenuation



Assume $r=0$ and d is unit step disturbance.

Only integral action : $C(s) = k_i/s$.

$$G_{dy}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{s\,P(s)}{s+P(s)k_i}$$

$$G_{dy}(0) = \frac{0}{P(0)k_i} = 0.$$

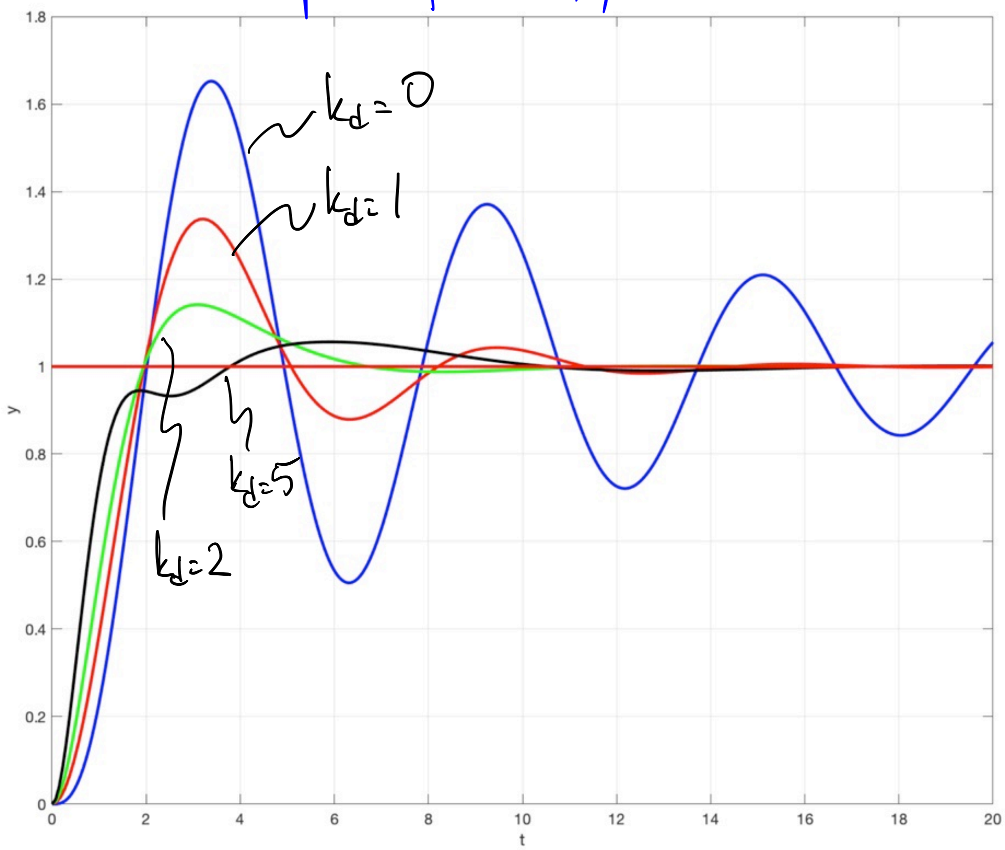
perfect disturbance rejection at steady-state.

Let us now add the derivative term:

$$C(s) = k_p + k_i/s + k_d s.$$

Check the response under unit step reference r:

Step response, y



Fixed:
$k_p = 2.5$
$k_i = 1.5$

With increasing k_d, the closed-loop system becomes more damped.

Example (effect of the k_d-term)

Let the reference be $r=0$. We will analyze the free response of the closed-loop system.

Open-loop system:

$$\ddot{y} + \alpha_1 \dot{y} + \alpha_2 y = u$$

Derivative control:

$$u = k_d \dot{e} = k_d(\dot{r} - \dot{y}) = -k_d \dot{y}$$

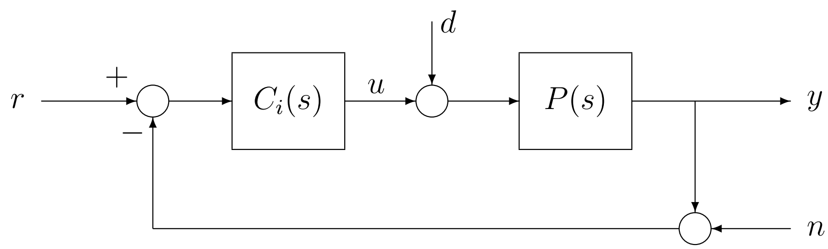
Closed-loop system: $\ddot{y} + (\alpha_1 + k_d)\dot{y} + \alpha_2 y = 0.$

α_2: unchanged $\Rightarrow$ ω_n: unchanged.

choose k_d to increase $\alpha_1 + k_d = 2\zeta\omega_n$.

From Midterm 2, Spring 2019

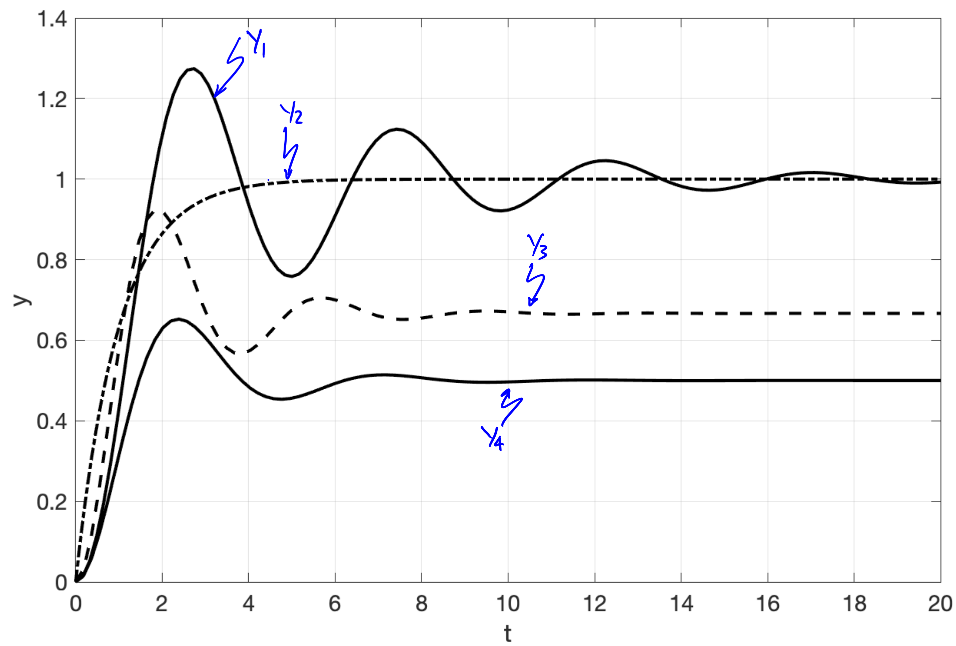
1. Let $P(s)$ be the transfer function for a stable, second-order linear system, and consider the feedback interconnection below, where $C_i(s)$ is the transfer function for a controller.



Consider four different controllers given by the following transfer functions:

- $C_1(s) = 1$
- $C_2(s) = 2$
- $C_3(s) = 1 + \frac{1}{s}$
- $C_4(s) = 1 + \frac{1}{s} + s$

The figure below shows the unit step responses $(Y_1, \ldots, Y_4)$ from r to y for the closed-loop system with these four different controllers. Match $C_1, \ldots, C_4$ to $Y_1, \ldots, Y_4$. Show the matching in the table in the next page, and briefly explain your reasoning.



Controller	Response
C_1	Y_4
C_2	Y_3
C_3	Y_1
C_4	Y_2

.

* Y_3 and Y_4 have steady-state errors. Hence they are for proportional-only controllers. Y_3 has smaller error; therefore, it's for the larger gain.

* Y_1 has more oscillations than Y_2. Hence, Y_2 includes the derivative term.